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Continuity of the Spectrum Function in convex cones in Ordered Banach Algebras

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Abstract: Let A be an ordered Banach algebra and C be a closed and convex cone in A. In this paper we define the C-continuity at a point $x_o \in C$ of the spectrum function $f: C \to \mathbb{C}$. We define some subalgebra of the OBA A on which the spectrum function $f(x) = \sigma(x)$ becomes continuous. We prove some results on subalgebra of A on which the spectrum function $f(x) = \sigma(x)$ becomes continuous when it is restricted to the subalgebra C. We also prove the continuity of the spectrum function in ordered Banach algebras with polynomial identities. Some examples will also be given.

KEYWORDS: Banach Algebras; Spectrum function; Algebra Cones; Convex Cone **MSC (2010)**: 47B60; 47A10; 47B65; 47L07; 46J10

1. Introduction

In the year 1951, J. D. Newburgh introduced the concept of spectral continuity in his paper [17]. He proved that in any Banach algebra, the spectrum function and therefore the spectral radius function is also upper semi-continuous. Since then, this topic has been widely studied by many mathematicians and several authors have made great contributions in this subject. Many researchers and authors provided different types of sufficient conditions for spectral continuity. Most of the important earlier results were presented by B. Aupetit, in the monograph [4]. In the year 1988, in her survey paper [6], L. Burlando, gave an extensive account of these and subsequent results up to that time and she also supplied several useful references. Many authors reveal the continuity of the restriction of the spectrum function and the spectral radius function to special subsets of certain Banach algebras. A sufficient condition for the continuity of spectral radius at a point of a Banach algebra is given by G. J. Murphy in paper [16]. Such a condition, as well as Newburgh's condition, involves only topological structure of spectrum.

In Banach algebra of all linear and continuous operators on a separable Hilbert space, the continuity points of the spectral radius function have been completely characterized by J. B. Conway and B. B. Morrel in paper [7]. In paper [14], S. Mouton studied about spectral continuity on positive elements in ordered Banach algebra. Some of the more recent papers on spectral continuity include papers [1], [3], [9], [13], [14], [15], and [16].

In this paper we study some conditions on the continuity of the spectrum function f in ordered Banach algebra and its subalgebras. The continuity of the spectrum function in ordered Banach algebras with polynomial identities will also be studied.

In section 2, we provide the definitions of elements in Banach algebras. In section 3, we define an algebra convex-cone C of a unital complex Banach algebra A and ordered Banach algebra (OBA). We also define spectrum function in ordered Banach algebra.

In section 4, we prove our main results of continuity of the spectrum function in convex cones C in ordered Banach algebra (A, C) and some other subalgebras.

2. Spectrum of an element in Banach Algebra

Throughout $\mathbb{C}$ will be the field of complex numbers and A will be a complex unital Banach algebra. An element a in a unital Banach algebra A is said to be invertible (or non-singular) in A if there exists some $z \in A$ such that $az = za = 1$. The set of all invertible elements in A forms a group under usual operations. The spectrum of an element $a \in A$ will be denoted by $\sigma(a)$ or $\sigma(a, A)$ and defined by $\sigma(a) = \{\alpha \in \mathbb{C}: a - \alpha \text{ is not invertible in } A\}$. The spectrum of an element of a complex

Banach algebra is non empty closed and compact subset of the set of complex numbers $\mathbb{C}$.

If $a \in A$ such that $\sigma(a) = 0$, then a is said to be quasinilpotent. The set of quasinilpotent elements of A will be denoted by $QN(A)$. An element $a \in A$ is said to be a radical element if $aA \subseteq QN(A)$. The set of all radical elements is then referred to as the radical of A and is denoted by $Rad(A)$. We can also define the radical of the Banach algebra A as $Rad(A) = \{x \in A: 1 - ax \in Inv(A)\ \forall a \in A\}$.

3. Ordered Banach Algebras

In section 3 of paper [18], we defined an algebra cone C of a complex Banach algebra A and it is shown that C induced on A an ordering that is compatible with the algebraic structure of A. Such a Banach algebra is called an ordered Banach algebra (OBA).

A nonempty subset C of A is a cone of A if $C + C \subseteq C$ and $\lambda C \subseteq C$ (for $0 \leq \lambda \in R$). If C satisfies $C \cap -C = \{0\}$, then C is called a proper cone. Any cone C of A induces an ordering $\leq$ on A such that $a \leq b$ if and only if $b - a \in C$ for $a, b \in A$. It is shown that this ordering is a partial ordering on A. Furthermore, C is proper if and only if this partial order has the additional property of being antisymmetric, i.e. if $a \leq b$ and $b \leq a$, then $a = b$. Considering the partial order that C induces, we find that $C = \{a \in A: a \geq 0\}$ and therefore we call the elements of C positive. A cone C of A is called an algebra cone of A, if $C.C \subseteq C$ and $1 \in C$.

Motivated by these concepts and from paper [1], we call a unital complex Banach algebra A an ordered Banach algebra (OBA) if A is partially ordered by a relation " $\leq$ " in such a manner that for every $a, b, c \in A$ and $\lambda \in \mathbb{C}$,

(1) $a, b \geq 0 \Rightarrow a + b \geq 0$,
(2) $a \geq 0, \lambda \geq 0 \Rightarrow \lambda a \geq 0$,
(3) $a, b \geq 0 \Rightarrow ab \geq 0$,
(4) $1 \geq 0$.

Therefore, if A is ordered by an algebra cone C, then A, or more specifically (A, C), is an OBA.

A cone C of A is called algebra convex-cone if it satisfies the following:

(i) $ab \in C$ for all $a, b \in C$ such that $0 \leq \lambda \leq 1$ implies $\lambda a + (1 - \lambda)b \in C$.

(ii) $1 \in C$, where 1 is the unit of A.

In the above definition, if for all $a, b \in C$ such that $ab = ba$, then we say that the cone is commutative convex cone.

4. Continuity of the Spectrum Function in convex cones

A well-known example about the discontinuity of the spectrum function is given by S. Kakutani, in the book [19], on pages 282-283 written by C.E. Rickart, also shows that in an ordered Banach algebra, the spectrum function is not always continuous on the algebra cone.

Following monograph [4], in order to measure the continuity of the spectrum, we introduce a distance on the compact subsets K_1, K_2 of the set of all complex numbers $\mathbb{C}$. This distance is called the Hausdorff distance and it is defined as

$$\Delta(K_1, K_2) = \max\{\sup_{z \in K_2} \operatorname{dist}(z, K_1), \sup_{z \in K_1} \operatorname{dist}(z, K_2)\}.$$

The spectrum function $x \to \sigma(x)$ or $x \to \sigma(x, A)$ is said to be continuous at $a \in A$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - a\| < \delta$ implies that the Hausdorff distance $\Delta(\sigma(x), \sigma(a)) < \epsilon$. It is well known from paper [17], that spectrum function $x \to \sigma(x)$ is upper semi-continuous on A, that is for every open set U containing $\sigma(x)$ there exists $\delta > 0$ such that $\|x - a\| < \delta$ implies $\sigma(a) \subset U$.

Now, we define the C-continuity at a point $x_o \in C$ of the spectrum function $f: C \to \mathbb{C}$, where C is closed and convex cone in Banach algebra A.

Definition 4.1. Let C be closed and convex-cone in an OBA A. The spectrum function $f(x) = \sigma(x)$ is C-continuous at a point $x_o \in C$, if $f(x_n) \to f(x_o)$ for each $\{x_n\} \subset C$ such that $\|x_n - x_o\| \to 0$.

The spectrum function $x \to \sigma(x)$ is C-continuous on the set E subset of C, if it is C-continuous at every point of the set E. If for a given $\epsilon > 0$, the number $\delta > 0$ is independent of $x_o \in E$, then the spectrum function $x \to \sigma(x)$ is uniformly continuous on E.

We also say that the spectrum function $f(x) = \sigma(x)$ is A-continuous at a point x_o (or on a set $E \subset A$) if f is A-continuous at a point x_o (or on a set $E \subset A$).

Now, we give some examples of continuity in Banach algebra.

Example 4.2. Let A be a commutative Banach algebra, the spectrum function $f(x) = \sigma(x)$ is continuous on A and hence $f(x) = \sigma(x)$ is C-continuous on the set E of subset of C and each subset E, where $E \subset C \subset A$.

Example 4.3. Let A be a Banach algebra, the spectrum function $f(x) = \sigma(x)$ is uniformly continuous on A if and only if the quotient algebra $A/Rad(A)$ is commutative (Chapter 2, [4]).

Example 4.4. Let C be a finite dimensional closed and convex-cone in an OBA (A, C), then the spectrum function $f(x) = \sigma(x)$ is continuous on C.

Definition 4.5. Let C be closed and convex-cone in an OBA A. For each closed subset $F \subset \mathbb{C}$, $F(C, A)$ is defined as $F(C, A) = \{x \in C: \sigma(x) \subset F\}$.

The following theorem is about C-continuity of the spectrum function $f(x) = \sigma(x)$ relative to the set $F(C, A)$.

Theorem 4.6. Let C be closed and convex-cone in an OBA A. The spectrum function $f(x) = \sigma(x)$ is C-continuous on C if and only if the set $F(C, A)$ is closed in C for each F closed in $\mathbb{C}$.

Proof: When $C = A$, then this theorem can be proved in a similar way as Theorem 1, Chapter 5 in [5]. For $C \neq A$, then the proof is as follow.

Suppose that $f(x) = \sigma(x)$ is C-continuous on C. Let $x_n \in F$ for some closed subset $F \subset \mathbb{C}$, therefore for $x_o \in F$, we have $\|x_n - x_o\| \to 0$. Since $f(x_n) \subset F$ and C-continuity of $f(x)$ implies that $f(x_n) \to f(x_o)$ it follows that $f(x_o) \subset F$. Thus $\sigma(x_n) = f(x_n) \subset F$ and $\sigma(x_o) = f(x_o) \subset F$. Hence, $F(C, A)$ is closed in C.

Conversely suppose that $F(C, A)$ is closed in C for each F closed in $\mathbb{C}$. To prove that f is C-continuous on C. On the contrary we suppose that that f is discontinuous at some point $x_o \in C$. Then there exist a sequence $x_n \in F$, $\lambda_o \in \sigma(x_o)$ and for some $\delta > 0$ such that $\|x_n - x_o\| \to 0$ and $\Delta(\lambda_o, \sigma(x_n)) \geq \delta$ for each $n \in N$. Take $F = \mathbb{C} - \{\lambda: |\lambda - \lambda_o| < \delta\}$, then for each $n \in N$ we have $f(x_n) \in F$ and hence $x_n \in F$, but $x_o \notin F$. Therefore F is not closed in $\mathbb{C}$. Hence f must be C-continuous.

Let C be closed and convex-cone in an OBA A. Let $H = \{h_s, s \in S\}$ be the set of all homomorphisms $h_s: C \to C_s$ acting from C into some algebra $C_s \coloneqq Imh_s$. Let Y be a closed subset of C, then we say that the set H generates a symbol for the set Y in OBA A if $\sigma(x, A) = \cup_{s \in S}\, \sigma(h_s(x), C_s)$ for every $x \in Y$.

We can also say that a set of mappings $\{g_s\}$, $g_s: Y \to Img_s$ generates a symbol for a set Y in OBA A if there exists a cone C, $(Y \subset C \subset A)$ and a set of homomorphisms $H = \{h_s\}$ of the OBA A such that H generates a symbol for the set Y in OBA A via the definition $\sigma(x, A) = \cup_{s \in S}\, \sigma(h_s(x), C_s)$ and $h_s|Y = g_s$.

Theorem 4.7. Let C be closed and convex-cone in an OBA A and Y be a closed subset of C. Suppose that $H = \{h_s, s \in S\}$ generates a symbol for Y in OBA A. If for every $s \in S$ the spectrum function $\sigma(h_s(x), C_s)$ is C_s-continuous on $h_s(Y)$, then $\sigma(x, A)$ is Y-continuous on Y.

Proof: Let F be a closed subset in $\mathbb{C}$, then by Theorem 4.6 $F(Y, A)$ is closed in Y. For each $n \in N$ there exist $x_n \in F(Y, A)$ such that $\|x_n - x_o\| \to 0$ implies $x_o \in F(Y, A)$. Since $\sigma(x_n, A) \subset F$ and $H = \{h_s, s \in S\}$ generates a symbol for Y in A. Thus, $\sigma(x, A) =$

$\cup_{s\in S}\,\sigma(h_s(x), C_s)$, therefore for every $s \in S$, $n \in N$ it follows that $\sigma(h(x_n), C_s) \subset F$. Hence,

$$\sigma(h(x_0), C_s) = \lim_{n\to\infty} \sigma(h(x_n), C_s) \subset F$$

for every $s \in S$. Thus,

$$\sigma(x_0, A) = \cup_{s\in S}\,\sigma(h_s(x_0), C_s) \subset F.$$

Therefore, we have $x_o \in F(Y, A)$. By using Theorem 4.6, we conclude that the spectrum function $\sigma(x, A)$ is Y-continuous on Y.

Theorem 4.8. Let C be a commutative convex cone in an OBA A, then the spectrum function $f(x) = \sigma(x, A)$ is C-continuous on C.

Proof: Let M be a maximal commutative subalgebra of A such that $C \subset M$. Since M is commutative algebra, therefore the spectrum function $\sigma(y, M)$ is continuous on M. Also M is inverse closed in OBA A i.e., $\sigma(y, M) = \sigma(y, A)$ for every $y \in M$. Thus the spectrum function $\sigma(y, A)$ is M-continuous on M and hence the spectrum function $\sigma(y, A)$ is also continuous on its subalgebra C. Therefore $f(x) = \sigma(x, A)$ is C-continuous on C.

Remark 4.9. Let A be an OBA. Let $f(x) = \sigma(x, A)$ is discontinuous at a point x_o. Suppose that M be the unital algebra generated by the element x_o. This gives that M is a commutative subalgebra of A for which the spectrum function $f(x) = \sigma(x, A)$ is discontinuous on M.

Let A be algebra of polynomials. Let $x_1, x_2, \dots, x_m \in A$ are non commuting variables and let F_m be a standard polynomial

$$F_m(x_1, x_2, \dots, x_m) = \sum_{\tau\in S_m} sign\ \tau x_{\tau(1)} x_{\tau(2)} \cdots x_{\tau(m)},$$

where τ runs through the symmetric group S_m.

The algebra A satisfies a standard polynomial identity of order m if there exists a nontrivial polynomial F_m such that $F_m(a_1, a_2, \dots, a_m) = 0$ for any elements $a_1, a_2, \dots, a_m \in A$ and we write it as $A \in Pi(m)$.

If $dim\ Im\ h_s \le n$ for each $s \in S$, then the algebra A is said to be with matrix symbol of order n and we write it as $A \in \pi_n$.
The following theorem form [12], describes all Banach algebras which admit a matrix symbol of order n.

Theorem 4.10. (Theorem 22.1, [12]) Let A be a Banach algebra. Then $A \in \pi_n$ if and only if $F_{2n}(a_1, a_2, \dots, a_{2n}) \in R(A)$ for every collection of elements $a_1, a_2, \dots, a_{2n} \in A$.

If $F_{2n}(a_1, a_2, \dots, a_{2n}) \in R(A)$ for every collection of elements $a_1, a_2, \dots, a_{2n} \in A$, then $A/R(A) \in F_{2n}$. By Theorem 21.1 [12], $A/R(A) \in \pi_n$ and hence, $A \in \pi_n$.

By considering these concepts, Theorem 4.10 can also be stated as

$$A \in \pi_n \quad \Leftrightarrow \quad A/R(A) \in Pi(2n).$$

Theorem 4.11. Let A be a Banach algebra. Suppose that $A/R(A) \in Pi(2n)$ for some $n \in N$. Then the spectrum function $\sigma(x, A)$ is continuous on A.

Proof: Let A be a Banach algebra and $A/R(A) \in Pi(2n)$ for some $n \in N$. Then by Theorem 4.10, the algebra A is with matrix symbol of order n and therefore, $dim\ Im\ h_s \le n$. Hence the spectrum function $\sigma(h_s(x), Im\ h_s)$ is continuous for each $s \in S$. Therefore from Theorem 4.7, it follows that the spectrum function $\sigma(x, A)$ is continuous on A.

5. Conclusions

In this paper, we defined the C-continuity at a point $x_o \in C$ of the spectrum function $f: C \to \mathbb{C}$ in convex cone in ordered Banach algebra A. We also defined some subalgebra of the OBA A on which the spectrum function $f(x) = \sigma(x, A)$ became continuous. We proved some results on subalgebra of A on which the spectrum function $f(x) = \sigma(x)$ became continuous when it is restricted to the subalgebra C. We also proved the continuity of the spectrum function in ordered Banach algebras with polynomial identities. Some examples are also given.

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